Lecture 4

Elliptic Operators: Solvability and Regularity

Let's again begin with an example, this time a pretty simple one:

$$u_{xx} = g(x) \qquad 0 < x < 1$$

$$u(0) = 0 = u(1)$$

Suppose first that $g(x)$ is continuous.

Let $$a = -\int_0^1 \int_0^y g(z)\,dz\,dy$$

(4.1) Then $$u(x) = ax + \int_0^x \int_0^y g(z)\,dz\,dy$$

is twice continuously differentiable

with

$$u'(x) = a + \int_0^x g(z)\,dz$$

$$u''(x) = g(x)$$

$$u(0) = 0$$

and $\quad u(1) = a + \int_0^1 \int_0^y g(z)\,dz\,dy = 0$

Suppose now that $g(x)$ is discontinuous (problems involving discontinuous terms arise naturally in ecological models if the environment changes drastically across some interface and we will want to consider such scenarios).

What about (4.1) in such a case? To be specific, let's take

$$g(x) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$$

Then $\quad a = -\int_0^1 \left[\int_0^y g(z)\,dz\right] dy$

$$= -\int_{1/2}^1 \int_{1/2}^y dz\,dy = -\int_{1/2}^1 (y - 1/2)\,dy$$

$$= -\left[1/2\, y^2 - 1/2\, y \right]\Big|_{1/2}^1$$

$$= -\left[1/2 - 1/2 - 1/8 + 1/4 \right] = -1/8$$

So for $x \in [0, \frac{1}{2}]$

$$u(x) = -\frac{1}{8}x$$

For $x \in [\frac{1}{2}, 1]$,

$$\int_0^x \int_0^y g(z)\,dz\,dy = \int_{1/2}^x \int_{1/2}^y dz\,dy$$

$$= \int_{1/2}^x (y - \tfrac{1}{2})\,dy$$

$$= \tfrac{1}{2}y^2 - \tfrac{1}{2}y \Big|_{1/2}^x$$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x + \tfrac{1}{8}$$

$$\Rightarrow u(x) = \tfrac{1}{2}x^2 - \tfrac{5}{8}x + \tfrac{1}{8}$$

Now $u'(x) = \begin{cases} -\frac{1}{8} & 0 \le x < \frac{1}{2} \\ x - \frac{5}{8} & \frac{1}{2} < x \le 1 \end{cases}$

Note that $\lim_{x \to 1/2} (x - \frac{5}{8}) = -\frac{1}{8} \Rightarrow$

$u'(x)$ is continuous on $[0, 1]$.

However, $u''(x) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 < x < 1 \end{cases}$.

Hence $u''(1/2)$ fails to exist and $u(x)$ satisfies the differential equation on $[0,1] - \{1/2\}$, but not throughout $[0,1]$. Here $u(x)$ satisfies the integrated version of the equation; i.e., (4.1), but not the original, since $u(x)$ is not smooth enough.

However $u''(x) - g(x) = 0$ if $x \neq 1/2$, so

$$\int_0^1 |u''(x) - g(x)|^2 \, dx = 0$$

Thus $u''(x) = g(x)$ in $L^2[0,1]$

This example leads us in a natural way to the notion of weak solutions

A standard way of defining weak solutions is based on the observation that if $f(x)$ and $g(x)$ are continuous functions so that

$$\int_0^1 h(x) f(x)\,dx = \int_0^1 h(x) g(x)\,dx$$

for all smooth functions $h(x)$, then

$$f(x) = g(x)$$

for all x. To see that such is the case, let $x_0 \in [0,1]$ be fixed. Choose a sequence of smooth functions $h_n(x)$ with $h_n(x) = 0$ outside $x_0 - 1/n < x < x_0 + 1/n$, $h_n(x) \geq 0$ and $\int_0^1 h_n(x) = 1$. Let $\varepsilon > 0$ be given. $\exists\, N \ni f(x_0) - \varepsilon < f(x) < f(x_0) + \varepsilon$ if $x \in (x_0 - 1/N, x_0 + 1/N)$. For $n > N$,

$$\int_0^1 h_n(x) f(x)\,dx = \int_{x_0 - 1/n}^{x_0 + 1/n} h_n(x) f(x) \geq \int_{x_0 - 1/n}^{x_0 + 1/n} h_n(x)\big(f(x_0) - \varepsilon\big)\,dx$$

$$= \big(f(x_0) - \varepsilon\big) \int_{x_0 - 1/n}^{x_0 + 1/n} h_n(x)\,dx = f(x_0) - \varepsilon.$$

Similarly, $\int_0^1 h_n(x)f(x)dx \leq f(x_0)+\varepsilon$.

Since ε is arbitrary, $\int_0^1 h_n(x)f(x) \to f(x_0)$.

The same argument shows $\int_0^1 h_n(x)g(x) \to g(x_0)$.

$\therefore f(x_0) = g(x_0)$.

Now $\int_0^1 h(x)f(x)dx = \int_0^1 h(x)g(x)dx$ could continue to hold even if $f(x)$ or $g(x)$ is discontinuous. But then we could not guarantee $f(x)=g(x)$ for all x.

We may now define weak derivatives, which provide a way of formulating weak solutions. Suppose $f(x)$ is differentiable and $h(x)$ is smooth with $h(0) = h(1) = 0$.

Then integration by parts $\Rightarrow$

$$\int_0^1 f'(x)h(x)dx = f(x)h(x)\Big|_0^1 - \int_0^1 f(x)h'(x)dx$$

$$= -\int_0^1 f(x)h'(x)\,dx.$$

So we can define a weak derivative of $f(x)$ to be a function $g(x)$ so that for any smooth $h(x)$ with $h(0) = 0 = h(1)$

(4.2) $$\int_0^1 g(x)\,h(x)\,dx = -\int_0^1 f(x)\,h'(x)\,dx$$

If $f'(x)$ exists as a continuous function, we can take $g(x) = f'(x)$. But (4.2) could hold in some cases when $f'(x)$ fails to exist everywhere. For example, take

$$f(x) = \begin{cases} -1/8 & 0 \le x < 1/2 \\ x - 5/8 & 1/2 \le x \le 1 \end{cases}$$

and

$$g(x) = \begin{cases} 0 & 0 \le x < 1/2 \\ 1 & 1/2 \le x \le 1 \end{cases}$$

Then $\int_0^1 g(x)h(x)\,dx = \int_{1/2}^1 h(x)\,dx$

and $\int_0^1 f(x)h'(x)\,dx = -\frac{1}{8}\int_0^{1/2} h'(x) + \int_{1/2}^1 (x-\frac{5}{8})h'(x)\,dx$

$$= -\frac{1}{8}h(1/2) + h(x)(x-5/8)\Big|_{1/2}^1 - \int_{1/2}^1 h(x)\,dx$$

$$= -\frac{1}{8}h(1/2) - h(1/2)\left[1/2 - 5/8\right] - \int_{1/2}^1 h(x)\,dx$$

$$= -\int_{1/2}^1 h(x)\,dx$$

So $\int_0^1 g(x)h(x)\,dx = -\int_0^1 f(x)h'(x)\,dx$ for all $h \in C'[0,1]$ with $h(0) = 0 = h(1)$

We will now interpret d^2/dx^2 as an operator on suitable spaces; i.e., as a function between Banach spaces. To do so, we need to specify the domain and define its action on said domain. The most natural or simplest choice here would be $C_0^2[0,1]$

with

$$\|u\|_{C^2[0,1]} = \sup_{x\in[0,1]} |u''(x)| + \sup_{x\in[0,1]} |u'(x)| + \sup_{x\in[0,1]} |u(x)|$$

Since a sequence of continuous functions that is Cauchy in the $\| \|_\infty$ norm is uniformly convergent to a continuous function, it follows from Ascoli-Arzeala that $C_0^2[0,1]$ is a Banach space.

Let $A = d^2/dx^2$. Then A maps $X = C_0^2[0,1]$ into $C[0,1]$. If $u, v \in X$, then

$$\|Au - Av\|_{C[0,1]}$$

$$= \|A(u-v)\|_{C[0,1]}$$

$$= \sup_{x\in[0,1]} |u''(x) - v''(x)| \le \|u-v\|_{C^2[0,1]}$$

So A is continuous.

For $g \in C[0,1]$, $u(x)$ given by (4.1)

is an element of X so that

$$Au(x) = g(x)$$

so $u(x) = A^{-1}g(x)$

Now $u(x) = \left(-\int_0^1\int_0^y g(z)\,dz\,dy\right)x + \int_0^x\int_0^y g(z)\,dz\,dy$

$u'(x) = \left(-\int_0^1\int_0^y g(z)\,dz\,dy\right) + \int_0^x g(z)\,dz$

$u''(x) = g(x)$

So it is evident that there is a constant C

(independent of g) so that

$$\|u\|_{C^2[0,1]} \leq C\|g\|_{C[0,1]}$$

So $A^{-1} : C[0,1] \to X$ is continuous.

In particular, A^{-1} maps bounded sets

in $C[0,1]$ into bounded sets in X.

Ascoli-Arzeala tells that any bounded set in X has compact closure in $C[0,1]$. Thus if we define E as the operator from X into $C[0,1]$ which acts as the identity operator on X; i.e.,

$$E u = u$$

then EA^{-1} maps $C[0,1]$ into $C[0,1]$ continuously, and in such a way that maps bounded sets into sets whose closures are compact.

The choice of $X \subset C^2[0,1]$ and $C[0,1]$ is not the only possibility. For example, we could also choose to work in the space Y of functions which are square integrable and which have weak first and second derivatives which are also square integrable with the norm

$$\|u\|_Y = \left[\int_0^1 \left(|u''(x)|^2 + |u'(x)|^2 + |u(x)|^2 \right) dx \right]^{1/2}$$

where the integrals are Lebesgue. Then Y is an example of a kind of Banach space known as a Sobolev space. A problem now arises with the boundary conditions, since $\|u\|_Y$ per se does not require $u(0)$ or $u(1)$ to even be defined. But if $u \in Y$ and $0 \le x_1 \le x_2 \le 1$, we can show

$$u(x_2) - u(x_1) = \int_{x_1}^{x_2} u'(x)\, dx$$

so that

(4.3) $$|u(x_2) - u(x_1)| \le \left(\int_{x_1}^{x_2} dx \right)^{1/2} \left(\int_{x_1}^{x_2} |u'(x)|^2 dx \right)^{1/2}$$

$$\le |x_2 - x_1|^{1/2} \|u\|_Y$$

In particular, if $u \in Y$, u is continuous, so it makes sense to speak of

$$Y_0 = \{ u \in Y \mid u(0) = u(1) = 0 \}$$

Now we can define an operator B with domain Y_0 by

$$B = \frac{d^2}{dx^2}$$

B and A are not the same operator, since they act on different spaces, even though Au and Bu represent the same function for u's for which both make sense.

B maps Y_0 into $L^2[0,1]$. It is clear that B is continuous. One can show that (4.1) defines an inverse operator B^{-1} that maps $L^2[0,1]$ continuously into Y_0. Indeed, it is also true that

$$E_0 : Y_0 \to L^2[0,1]$$

given by $E_0 u = u$ maps bounded sets in Y_0 into sets with compact closure in $L^2[0,1]$ (harder to show). So $E_0 B^{-1}$ has the same type of mapping properties on $L^2[0,1]$ as $E_0 A^{-1}$ has on $C[0,1]$.

Why more than one formulation?

Well, functions in $C[0,1]$ are continuous, so will have maxima and minima, for example, which might be useful. On the other hand, the norm in $L^2[0,1]$ arises from the inner product

$$\langle u, v \rangle = \int_0^1 u(x) v(x)\, dx$$

and inner products are often useful.

If we use Lebesgue integrals, $L^2[0,1]$ is a Banach space, as in Royden (1968) or Rudin (1966). So we always think of Lebesgue integrals here. (Note: If the Riemann integral exists, it is the same as the Lebesgue.)

Higher Space Dimensions

When we try to formulate appropriate state spaces for reaction-diffusion models in more than one space dimension, we encounter some more technical issues. (i) In one-space dimension, the boundary of an interval is always two points, but in two or three dimensions it can be very complicated.

We usually impose conditions on the boundaries, typically requiring them to be smooth curves or surfaces.

(ii) Another problem is that the elliptic operators which occur in our models do not have all the mapping properties we want in $C^2(\bar{\Omega})$.

(iii) The formula (4.3) which guarantees that functions in Y are continuous is not valid in exactly the same way in higher dimensions. Continuity is a highly desirable property, so sometimes we will work in spaces whose norms are defined via integrals but whose elements are continuous functions. To do this, we may use an analogue to Y_0 constructed from the space $L^p(\Omega)$, whose norm is $\|u\|_p = \left(\int_\Omega |u(x)|^p dx\right)^{1/p}$, where $p \geq 1$ (typically much larger than 2) (Adams (1975), Gilbarg and Trudinger (1977), Nash (1958) DeGiorgi (1957))

Back to the second issue. An operator L of the form (1.3) will be continuous as a map from $C^2(\bar{\Omega})$ into $C(\bar{\Omega})$, but now the inverse or solution operator L^{-1} for $Lu = f$ under suitable boundary conditions is not in general continuous as a map from $C(\bar{\Omega})$ to $C^2(\bar{\Omega})$. To get around this problem, we employ a notion of continuity for functions which is stronger, Hölder continuity.

Let $x_0 \in \mathbb{R}^n$. Let $\alpha \in (0,1]$. $f : \mathbb{R}^n \to \mathbb{R}$ is Hölder continuous with exponent α at x_0 if there is a neighborhood of x_0 and a constant C so that

$$|f(x) - f(x_0)| \le C|x - x_0|^\alpha$$

for all x in the neighborhood. (When $\alpha = 1$, Hölder continuity is called Lipschitz continuity.)

A function $f(x)$ is Hölder continuous of exponent α on $\overline{\Omega}$ if

$$[f]_\alpha = \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}$$

is finite. We use $C^\alpha(\overline{\Omega})$ to denote the space of such functions with norm

(4.4) $$\|u\|_\alpha = \|u\|_{0,\alpha} = \sup_{x\in\overline{\Omega}} |u(x)| + \sup_{\substack{x,y\in\overline{\Omega} \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$$

$$= \|u\|_\infty + [u]_\alpha$$

Now to describe partial derivatives of a function

u defined on a subset of $\mathbb{R}^n$, it is common practice to use the multi-index notation

$$\beta = (\beta_1, \ldots, \beta_n)$$

where $\beta_1, \ldots, \beta_n$ are nonnegative integers, to define the order of β, denoted $|\beta|$ by

$$|\beta| = \sum_{i=1}^{n} \beta_i ,$$

and to write

$$\frac{\partial^{|\beta|} u}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \cdots \partial x_n^{\beta_n}} \quad \text{as} \quad \partial^\beta u .$$

The space of functions on $\bar{\Omega}$ which have spatial derivatives up to order k that are continuous on $\bar{\Omega}$, with the derivatives of order k being Hölder continuous with exponent α, is denoted

$$C^{k,\alpha}(\bar{\Omega}) \quad \text{or} \quad C^{k+\alpha}(\bar{\Omega})$$

The norm of $u \in C^{k,\alpha}(\bar{\Omega}) = C^{k+\alpha}(\bar{\Omega})$ is

(4.5) $$\|u\|_{k+\alpha} = \sup_{x\in\bar{\Omega}} |u(x)| + \sum_{|\beta|\le k} \sup_{x\in\bar{\Omega}} |\partial^\beta u(x)| + \sum_{|\beta|=k} [\partial^\beta u]_\alpha$$

$C^{k+\alpha}(\bar{\Omega})$ is a Banach space under suitable assumptions on the domain (Gilbarg and Trudinger (1977), Friedman (1976)).

When we use $C^\alpha(\bar{\Omega})$, $L^{-1}u \in C^{2+\alpha}(\bar{\Omega})$, so we will usually start out with L as a map from $C^{2+\alpha}(\bar{\Omega})$ into $C^\alpha(\bar{\Omega})$. (See Gilbarg and Trudinger for more detail.)

We need conditions on the boundary that allow us to make changes of coordinates for our operators that map regions of $\bar{\Omega}$ to hyperplanes (i.e. "flatten them out"). When the differential operator is written in terms of the new coordinates derivatives of the functions defining the coordinate change occur in the coefficients by the chain rule. So the theory that has been developed requires that the changes of coordinates are smooth enough that the coefficients of the operator under consideration are well-defined and sufficiently smooth in the new coordinate system.

A change of coordinates is a function ϕ which maps an open set in $\mathbb{R}^n$ into another open set in $\mathbb{R}^n$ in such a way that ϕ has an inverse function ϕ^{-1} with ϕ, ϕ^{-1} at least continuous.

we say that the boundary $\partial\Omega$ of a domain $\Omega \subseteq \mathbb{R}^n$ is of class $C^{k+\alpha}$ if for each point $x_0 \in \partial\Omega$ there is a neighborhood U containing x_0 and a change of coordinates ϕ on U so that

(a) If $y = (y_1, \ldots, y_n) = \phi(x)$, the image of $\partial\Omega \cap U$ under ϕ lies in

$\{y \in \mathbb{R}^n \mid y_n = 0\}$ while the image of $\Omega \cap U$ lies in the half-space

$\{y \in \mathbb{R}^n \mid y_n > 0\}$

and

(b) Each of the coordinate functions $y_1(x), \ldots,$

$y_n(x)$ defining ϕ belongs to $C^{k+\alpha}(\bar{U})$ and each of the functions $x_1(y), \ldots, x_n(y)$ defining ϕ^{-1} belongs to $C^{k+\alpha}(\phi(\bar{U}))$, where $\phi(\bar{U})$ is the image of $\bar{U}$ under ϕ.

<u>Theorem 4.1</u> (CC Thm 1.1) Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with $\partial\Omega$ of class $C^{2+\alpha}$ for some $\alpha \in (0,1)$. Suppose that the coefficients of the operator L in (1.3) belong to $C^{\alpha}(\bar{\Omega})$, L is strongly elliptic and $c(x) \leq 0$. Then if $f \in C^{\alpha}(\Omega)$ and $g \in C^{2+\alpha}(\bar{\Omega})$, the problem

$$Lu = f(x) \quad \text{in } \Omega$$

(Dirichlet)

$$u = g(x) \quad \text{on } \partial\Omega$$

has a unique solution $u \in C^{2+\alpha}(\bar{\Omega})$ with

$$\|u\|_{2+\alpha} \le C\left(\|f\|_{\alpha} + \|g\|_{2+\alpha}\right)$$

for a constant C that is independent of f and g.

Note: (i) version of Theorem 6.14 in Gilbarg and Trudinger (1977). The boundary condition is formulated in terms of $g \in C^{2+\alpha}(\bar{\Omega})$.

(ii) If we started with $g_0 \in C^{2+\alpha}(\partial\Omega)$, since we assume $\partial\Omega$ is of class $C^{2+\alpha}$, we could extend g_0 to $g \in C^{2+\alpha}(\Omega)$ in such a way that $\|g\|_{C^{2+\alpha}(\bar{\Omega})} \le C\|g_0\|_{C^{2+\alpha}(\partial\Omega)}$

where C is independent of g_0.

(iii) $g \equiv 0$ is included.

Theorem 4.2 (CC Thm 1.2). Let Ω, L and f be as in Theorem 4.1. Suppose $g(x)$, $\gamma(x)$, $\beta(x)$ $\in C^{1+\alpha}(\partial\Omega)$ with $\beta(x) > 0$ and $\gamma(x) \geq 0$.

Suppose $c(x) \leq 0$ and that either $c(x) \not\equiv 0$ or $\gamma(x) \not\equiv 0$.

Let $\partial/\partial\eta$ denote the outward normal derivative on $\partial\Omega$. Then the problem

$$Lu = f(x) \quad \text{on } \Omega$$

(Neumann)

$$\gamma(x)u + \beta(x)\frac{\partial u}{\partial \eta} = g(x) \quad \text{on } \partial\Omega$$

has a unique solution $u \in C^{2+\alpha}(\bar{\Omega})$ and there is a constant C independent of f and g so that

$$\|u\|_{2+\alpha} \leq C\left(\|f\|_{\alpha} + \|g\|_{1+\alpha}\right)$$

where $\|g\|_{1+\alpha}$ is the norm of g in $C^{1+\alpha}(\partial\Omega)$.

Note: If $c(x) \equiv 0 \equiv \gamma(x)$, the boundary operator reduces to

$$\frac{\partial u}{\partial \eta} = \frac{g(x)}{\beta(x)}.$$

In this case, the boundary value problem may not have a solution and may have infinitely many solutions when it is solvable. To this end, let Ω be smooth enough to apply the Divergence Theorem and consider

$$\nabla^2 u = \Delta u = f \quad \text{in } \Omega$$

$$\nabla u \cdot \eta = 0 \quad \text{on } \partial\Omega$$

Then if there is a solution,

$$\int_\Omega f(x)\,dx = \int_\Omega \nabla\cdot(\nabla u)\,dx$$

$$= \int_{\partial\Omega} (\nabla u)\cdot\eta\,dx = 0$$

Hence if $\int_\Omega f(x)\,dx \neq 0$, the boundary value problem is not solvable. Moreover, if u is a solution, then if $v = u+k$, where k is a constant, then

$$\Delta v = \Delta(u+k)$$

$$= \Delta u + \Delta k = f + 0 = f$$

in Ω, with

$$\nabla v\cdot\eta = \nabla(u+k)\cdot\eta$$

$$= \nabla u\cdot\eta = 0$$

on $\partial\Omega$, so that the problem has infinitely many solutions.

Theorem 4.3 (CC Thm 1.3). Let L, Ω, γ and β be as in Thm 4.1 and Thm 4.2, except that we no longer insist that $c \leq 0$ or $\gamma \geq 0$. Then each of the two proceeding boundary value problems satisfies one or the other of the following:

(a) The homogeneous problem with $f = 0$, $g = 0$ has only the trivial solution $u = 0$ and the inhomogeneous problem has a unique solution $u \in C^{2+\alpha}(\bar{\Omega})$ for each $f \in C^{\alpha}(\bar{\Omega})$ and $g \in C^{2+\alpha}(\bar{\Omega})$ (Dirichlet) or $g \in C^{1+\alpha}(\partial\Omega)$ (Neumann).

(b) The homogeneous problem has nontrivial solutions; in that case, the null space of L

is a finite dimensional subspace of $C^{2+\alpha}(\bar{\Omega})$.

Note: (i) Version of Theorem 6.15 in Gilbarg and Trudinger (1977). It is a special case of what is known as the Fredholm Alternative.

(ii) In the case of homogeneous boundary conditions, under the hypotheses of Thms 4.1 and 4.2, L has an inverse L^{-1} which is compact from $C^{\alpha}(\bar{\Omega})$ into itself. More precisely, $L^{-1} : C^{\alpha}(\bar{\Omega}) \to C^{2+\alpha}(\bar{\Omega})$ is continuous, $C^{2+\alpha}(\bar{\Omega})$ embeds compactly into $C^{2}(\bar{\Omega})$ by Ascoli-Arzeala (Hölder condition gives us the needed equicontinuity for second partial derivatives.)

$C^2(\bar\Omega)$ embeds in $C^\alpha(\bar\Omega)$ by the Mean Value Theorem, so that if $E : C^{2+\alpha}(\bar\Omega) \to C^\alpha(\bar\Omega)$ is given by

$$Eu = u,$$

the map $EL^{-1} : C^\alpha(\bar\Omega) \to C^\alpha(\bar\Omega)$ is compact since L^{-1} is continuous and E is compact.

What if the right hand side f is continuous but not Hölder continuous, or even discontinuous? We will need an appropriate notion of weak derivative and an appropriate space in which to work. To motivate our definition, suppose that u, v are smooth in Ω and that v vanishes on $\partial\Omega$. Let $\vec{e}_k$ denote the unit vector in the x_k-direction and $\vec{\eta}$ the outward unit normal to Ω. Then

$$0 = \int_{\partial\Omega} (uv\vec{e}_k)\cdot\vec{\eta}\,dS = \int_\Omega \nabla\cdot(uv\vec{e}_k)\,dx = \int_\Omega \left(\frac{\partial u}{\partial x_k} v + u\frac{\partial v}{\partial x_k}\right)dx$$

$$\Rightarrow \int_\Omega \frac{\partial u}{\partial x_k} v\,dx = -\int_\Omega u\frac{\partial v}{\partial x_k}\,dx$$

So we say that w is the weak derivative of u with respect to x_k on Ω if

(4.6) $$\int_\Omega w v \, dx = -\int_\Omega u \frac{\partial v}{\partial x_k} dx$$

for all smooth v which vanish outside a compact subset of Ω. Higher weak derivatives are defined via iterations of (4.6); e.g. $w = \frac{\partial^2 u}{\partial x_k \partial x_j}$ weakly would require

$$\int_\Omega w v \, dx = -\int_\Omega \left(\frac{\partial u}{\partial x_j}\right)\left(\frac{\partial v}{\partial x_k}\right) dx = (-1)^2 \int_\Omega u \frac{\partial^2 v}{\partial x_j \partial x_k} = (-1)^2 \int_\Omega u \frac{\partial^2 v}{\partial x_k \partial x_j}$$

$$= \int_\Omega u \frac{\partial^2 v}{\partial x_k \partial x_j}$$

where v vanishes outside a compact subset of Ω and is smooth enough.

The spaces that we need will be constructed from weak derivatives in the setting of the spaces $L^p(\Omega)$ of functions on Ω whose pth power is absolutely integrable.

in the sense of Lebesgue (Royden (1968), Rudin (1976)).

A few brief comments: Lebesgue vs. Riemann

A Riemann integral $\int_{\Omega} f(x)dx$ is based upon approximations via so-called Riemann sums. Here the domain Ω of the function is subdivided into smaller units, a sample value of f taken for each subunit, and a sum is formed, each summand being the sample value times the measure (length, area, volume, etc) of the corresponding subunit.

For Lebesgue, the range is subdivided and the "measure" of the set of points that f maps into each subdivision is taken. One then approximates via a sum; where now each summand is the product of a sample value in the subdivision of the range and the "measure" of its inverse image under f.

The advantage of the second approach over the first is that convergence of Riemann sums requires f to be approximately constant on subunits of the domain as the process goes forward. If f is badly enough discontinuous that is not going to be the case, even if the set of points "causing" the discontinuity is in a sense "small". This is not a problem in the second way of approaching or constructing an integral. But if Riemann exists, so does Lebesgue.

As a result, if we want to interchange integration and other limiting processes, we do much better in the Lebesgue context. For $1 \le p < \infty$, we set $L^p(\Omega)$ as the set of all functions f on Ω for which $|f|^p$ is integrable in the Lebesgue sense and set

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{1/p}$$

$L^p(\Omega)$ is a Banach space under this norm. (Technically, elements of $L^p(\Omega)$ are equivalence classes of functions that agree off a set of "measure zero"). When $p=\infty$, $L^\infty(\Omega)$ consists of measurable functions which are bounded except perhaps on a set of measure zero. Then the norm on $L^\infty(\Omega)$ is given by

$$\|f\|_\infty = \inf\left\{ \sup_\Omega |g(x)| : g \text{ is measurable on } \Omega \text{ and } g=f \text{ except possibly on a set of measure } 0 \right\}$$

In our context, if $f \in C(\bar\Omega)$, $\|f\|_\infty = \sup_{\bar\Omega} |f|$, so that $C(\bar\Omega)$ is a closed subspace of $L^\infty(\bar\Omega)$.

$C(\bar\Omega)$ is a subspace of $L^p(\bar\Omega)$, $1 \le p < \infty$, but in general not a closed subspace thereof, as we have already seen.

Now let $W^{k,p}(\Omega)$ denote the space of functions whose weak derivatives of order up to k belong to $L^p(\Omega)$. We define the norm in $W^{k,p}(\Omega)$, $\|\cdot\|_{k,p}$

by $\|f\|_{k,p} = \sum_{|\beta|\le k} \|\partial_\beta f\|_p$ (or $\left(\sum_{|\beta|\le k} \|\partial_\beta f\|_p^p\right)^{1/p}$)

where $\beta = (\beta_1, \dots, \beta_n)$, β_i nonnegative integer,

$|\beta| = \sum_{i=1}^n \beta_i$ and $\partial_\beta f = \partial^{|\beta|} f / \partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}$

in the weak sense

$W^{k,p}(\Omega)$ is a Sobolev space and is Banach. (Adams 1975, Gilbarg and Trudinger 1997)

An alternate way to define these spaces is to complete $C^k(\bar{\Omega})$ or $C^{k+\alpha}(\bar{\Omega})$ with respect to the $\|\cdot\|_{k,p}$ norm.

A particular Sobolev space that we will use in our discussion of eigenvalues for elliptic operators is $W^{1,2}(\Omega)$, as its norm can be viewed as arising from the inner product

$$\langle u, v \rangle = \int_\Omega [\nabla u \cdot \nabla v + uv]\, dx$$

In some cases we will also want to work in the subspace $W_0^{1,2}(\Omega)$ of $W^{1,2}(\Omega)$ which is obtained by taking the completion relative to the 1,2 norm of the space of continuously differentiable functions that vanish outside a compact subset of Ω (so that in particular they vanish on $\partial\Omega$).

Since functions in Sobolev space need not be continuous on $\bar{\Omega}$, ∴ care must be taken in formulating boundary conditions on them. For functions $f(x) \in C(\bar{\Omega})$ the mapping from f to the restriction of f to $\partial\Omega$ maps $C(\bar{\Omega})$ continuously to $C(\partial\Omega)$. In Sobolev spaces, we employ the notion of trace. If Y is a space of functions on $\partial\Omega$, the function $u \in W^{m,p}(\Omega)$ has trace $y \in Y$ if for any sequence $\{u_n\}$ of infinitely differentiable functions on $\bar{\Omega}$ which converge to u in the norm of $W^{m,p}(\Omega)$, the functions obtained by restricting u_n to $\partial\Omega$ converge to y in the norm of Y, with $\|y\|_Y \leq C\|u\|_{m,p}$, where the constant C is independent of u.

Well, when will this work? The following result comes from Adams (1975).

Lemma 4.4. (CC Lemma 1.4). Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with $\partial\Omega$ of class $C^{2+\alpha}$. Then:

(i) For $p \in (1,\infty)$, functions in $W^{2,p}(\Omega)$ have traces in $W^{1,p}(\partial\Omega)$.

(ii) For $m = 1, 2$, $p < n/m$, functions in $W^{m,p}(\Omega)$ have traces in $L^q(\partial\Omega)$, $q \in \left[p, \frac{(n-1)p}{(n-mp)}\right]$

(iii) For $m = 1, 2$, $p \geq n/m$, functions in $W^{m,p}(\Omega)$ have traces in $L^q(\partial\Omega)$, $q \in [p,\infty)$

<u>Theorem 4.5</u> (CC Thm 1.5) Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with $\partial\Omega$ of class $C^{2+\alpha}$. Suppose the coefficients of L in (1.3) belong to $C^\alpha(\bar{\Omega})$, L is strongly uniformly elliptic and $c(x) \leq 0$.

Let $p \in (1, \infty)$. If $f \in L^p(\Omega)$ and $g \in W^{2,p}(\Omega)$, the problem

$$Lu = f(x) \quad \text{in } \Omega$$

$$u = g(x) \quad \text{on } \partial\Omega$$

has a unique solution $u \in W^{2,p}(\Omega)$ satisfying

$$\|u\|_{2,p} \leq C\left(\|f\|_{0,p} + \|g\|_{2,p}\right).$$

where the constant C is independent of f and g.

<u>Theorem 4.6</u> (CC Thm 1.6) Suppose Ω and L satisfy the hypotheses of Theorem 4.5. Suppose that c, β and γ satisfy the hypotheses of Theorem 4.2 and that $f \in L^p(\Omega)$ and $g \in W^{1,p}(\Omega)$. Then the problem

$$Lu = f(x) \quad \text{in } \Omega$$

$$\gamma(x)\, u + \beta(x) \frac{\partial u}{\partial \eta} = g(x) \quad \text{on } \partial\Omega$$

has a unique solution $u \in W^{2,p}(\Omega)$, with

$$\|u\|_{2,p} \le C\left(\|f\|_{0,p} + \|g\|_{1,p}\right),$$ where C

is independent of f and g.

Theorem 4.7 (CC Thm 1.7, partial) Suppose that $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with $\partial\Omega$ $C^{2+\alpha}$. Then if $mp > n$, $W^{m+k,p}(\Omega)$ embeds compactly in $C^k(\bar{\Omega})$.

Moreover, if $(m-1)p \le n < mp$ and $\alpha \in \left(0, m - \left(\frac{n}{p}\right)\right) \subseteq (0,1)$, then $W^{m+k,p}(\Omega)$ embeds compactly in $C^{k+\alpha}(\bar{\Omega})$.
(part of Rellich-Kondrachov; see Adams 1975)

Corollary 4.8. $W^{2,p}(\Omega)$ embeds compactly in $C^{1+\alpha}(\bar{\Omega})$ if $p > n$, and $\alpha \in \left(0, 1 - \left(\frac{n}{p}\right)\right)$

Pf: Let $m = 1$ and $k = 1$. If $n < p$, $0 = (m-1)p \le n < p = mp$ and $m - \left(\frac{n}{p}\right) = 1 - n/p$.

The way we typically use these results is to formulate the solution operator for L as a compact operator on $C(\bar{\Omega})$ or on $C^{1+\alpha}(\bar{\Omega})$.

Theorem 4.9 (CC Thm 1.8) Suppose L in (1.3) is strongly uniformly elliptic, with coefficients in $C^\alpha(\bar{\Omega})$ and $c \leq 0$. Suppose $\Omega \subseteq \mathbb{R}^n$ is a bounded domain with $\partial\Omega$ of class $C^{2+\alpha}$. Then the solution operator L^{-1} for the problem

$$Lu = f \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega$$

can be extended to $C(\bar{\Omega})$ from $C^\alpha(\bar{\Omega})$ (or restricted from $L^p(\Omega)$ to $C(\bar{\Omega})$) to define a compact operator from $C(\bar{\Omega})$ into $C(\bar{\Omega})$. Similarly, L^{-1} can be restricted from $C^\alpha(\bar{\Omega})$ to $C^{1+\alpha}(\bar{\Omega})$.

Proof: Suppose $f \in C(\bar{\Omega})$. Then $f \in L^p(\Omega)$ for any p. Choose $p > n$. Thm 4.5 $\Rightarrow$ the solution operator $L^{-1} : L^p(\Omega) \to W^{2,p}(\Omega)$ is continuous; i.e.

$$\|L^{-1} f\|_{2,p} \leq C \|f\|_{0,p}$$

for all $f \in L^p(\Omega)$, where C is independent of f.

Now, since $f \in C(\bar{\Omega})$, $\|f\|_{0,p} \leq C_1 \|f\|_\infty$

where $\|f\|_\infty$ is the sup-norm of f. So

$$\|L^{-1}f\|_{2,p} \le CC_1 \|f\|_\infty$$

Let $E : W^{2,p}(\Omega) \to C(\bar{\Omega})$ be the map

$$Eu = u.$$

Corollary 4.8 $\Rightarrow$ the embedding $E_1 : W^{2,p}(\Omega) \to C^{1+\alpha}(\bar{\Omega})$ is compact. Indeed so is the embedding $E_2 : C^{1+\alpha}(\bar{\Omega}) \to C(\bar{\Omega})$. Since $E = E_2 E_1$, E is compact.

Thus EL^{-1} defines a solution operator which is compact as an operator from $C(\bar{\Omega})$ to $C(\bar{\Omega})$.

The case of $C^{1+\alpha}(\bar{\Omega})$ is even easier and is left as an exercise.

What does having the solution operator be compact buy us? A compact operator A from a Banach space into itself has a countable set of eigenvalues, that can accumulate only at 0.

The multiplicity of all nonzero eigenvalues is finite; i.e.,

$$\dim N(A - \sigma I) < \infty \quad \text{for all } \sigma \neq 0.$$

Compact linear operators are continuous, hence bounded in the sense that

$$\|A\| = \sup \left\{ \frac{\|Ax\|}{\|x\|} ; x \in X, x \neq 0 \right\} < \infty$$

If σ is an eigenvalue, $Ax = \sigma x$ for some $x \neq 0$.

Hence $|\sigma| = \frac{\|Ax\|}{\|x\|} \leq \|A\|$. So the set of eigenvalues of a compact operator is bounded.

Suppose now L^{-1} exists as a compact operator on a space X. Let μ be a non-zero eigenvalue of L^{-1}. Then we have

$$L^{-1} x = \mu x$$

so we want to say

$$Lx = (1/\mu) x$$

But let's be careful out there. We are right, but let's see why. In our context, $A = EL^{-1}$ where $L^{-1} : X \to Y$ is continuous and $E : Y \to X$ is a compact embedding. So to apply L to x we need to know that $x \in Y$.

We get

$$x = \frac{1}{\mu} E L^{-1} x = E\left(\frac{1}{\mu} L^{-1} x\right)$$

and $\frac{1}{\mu} L^{-1} x \in Y$.

But since $\frac{1}{\mu} L^{-1} x \in Y$, $E\left(\frac{1}{\mu} L^{-1} x\right) = \frac{1}{\mu} L^{-1} x$.

Thus $x = \frac{1}{\mu} L^{-1} x \in Y$ and thus we can apply L to both sides with impunity.